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RESEARCH MEMORANDUM

MATHEMATICAL RELATIONSHIPS BETWEEN PAY GRADE STRUCTURE, LONGEVITY, AND PROMOTION POLICY

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A Division of CNA Hudson Institute

CENTER FOR NAVAL ANALYSES

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During the derivation of manpower requirements and personnel management policies, it is quite possible to specify a pay grade profile, longevity distribution, and promotion policy that are inconsistent and cannot be simultaneously satisfied. This research memorandum describes precise conditions under which particular specifications of pay grade structure, longevity, and promotion policy lead to an executable force strucutre. In particular, it is shown that a too stringent "up or out" policy leads to an unexecutable force strucuture.

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1. Enclosure (1) is forwarded as a matter of possible interest.

2. This Research Memorandum determines situations under which possible specifications of pay grade structure, longevity, and promotion policy are inconsistent and cannot be simultaneously satisfied. In particular, it is shown that a too stringent "up or out" policy leads to an unexecutable force structure.

Lewis R. Cabe
Director
Manpower and Training Program

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MATHEMATICAL RELATIONSHIPS BETWEEN PAY GRADE STRUCTURE, LONGEVITY, AND PROMOTION POLICY

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ABSTRACT

During the derivation of manpower requirements and personnel management policies, it is quite possible to specify a pay grade profile, longevity distribution, and promotion policy that are inconsistent and cannot be simultaneously satisfied. This research memorandum describes precise conditions under which particular specifications of pay grade structure, longevity, and promotion policy lead to an executable force structure. In particular, it is shown that a too stringent "up or out" policy leads to an unexecutable force structure.

EXECUTIVE SUMMARY

Navy manpower requirements describe the number of personnel required to fulfill Navy missions. These manpower requirements are described by a variety of terms, including skill, pay grade, and length of service. Navy personnel managers institute numerous policies in order to attain and then to maintain the required number of personnel. In particular, compensation levels are systematically varied to obtain desired continuation rates, and promotion policy may be altered to attain desired numbers of personnel in each pay grade and longevity profiles within pay grades.

The interactions between personnel management policies and force structure are many and complex. For example, suppose there is an increase in requirements for E-5 personnel, and one needs to estimate the effect of this increment on force management. One may immediately observe that an increase in E-5 requirements will raise promotion opportunities for E-4 personnel and lower promotion opportunities for E-5 personnel. These changes in promotion opportunity will affect continuation rates, which in turn will cause variations in longevity profiles.

It is not apparent that a desired force structure can always be obtained. The above example exhibits some of the issues that need to be addressed when considering whether a desired force structure is practical. The situation is exacerbated by the fact that personnel management decisions are often made in isolation and not in the context of overall force management. This research memorandum addresses those issues by considering the interactions between certain force structure variables. In particular, pay grade structure, longevity profiles, and promotion policies are considered. Those situations are identified in which combinations of pay grade structure, longevity profiles, and promotion policy lead to unexecutable force structures.

It was concluded that stringent "up or out" policies are precisely the situations in which desired force structure becomes unobtainable. This is because the ability to successfully manage personnel can be shown to be directly related to the amount of permissible overlap in longevity between different pay grades. For example, suppose E-4 personnel are not allowed to have more than 8 years of service, and promotion opportunity is such that only a few E-5 personnel have less than 8 years of service. Then, it is unlikely that it would be possible to both attain and maintain desired force structure.

The exact point at which "up or out" policies become too stringent is not readily defined. In any particular situation, fluctuations in continuation behavior may make a force structure attainable one year but unobtainable the next. However, the less the allowable overlap in longevity between pay grades, the more likely the force structure will be unexecutable.

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INTRODUCTION

Navy manpower requirements describe the number of personnel required for the fulfillment of Navy missions. These manpower requirements are described in terms of skill, pay grade, length of service, etc. A common way to view requirements is as a length-of-service (LOS)-by-pay-grade matrix as displayed in table 1.

TABLE 1

HYPOTHETICAL ENLISTED FORCE STRUCTURE

	Pay grade					
LOS	E-1	E-2	E-3	...	E-9	Total
1	35,000	33,000	0		0	68,000
2	0	11,000	51,000		0	62,000
3	0	0	25,000		0	57,000
.	.	.	.		.	.
.	.	.	.		.	.
.	.	.	.		.	.
30+	0	0	0		150	900
Total	35,000	44,000	90,000		4,500	525,000

The row totals in the above table represent the total number of personnel in each LOS cell, and the column totals display the total number of personnel in each pay grade. The matrix entries show how the personnel are spread across the various combinations of pay grade and length of service. The many "zero" entries reflect the fact that personnel in a particular pay grade must have length of service within specific bounds. So, for example, there are no E-9s with only four years of service, etc.

Table 1 results from three distinct aspects of manpower planning. First, the column totals show the distribution of personnel that may be derived from requirements documents (SMDs, SQMDs, etc.). Secondly, the rules that govern promotion help to determine which entries in the matrix may be nonzero. Finally, if the force structure is going to be maintained from one year to the next, then the row totals will reflect continuation rates. For example, if there are 62,000 personnel with LOS = 2 and 57,000 personnel with LOS = 3, then a continuation rate of 57,000/62,000 = 0.92 between LOS cells 2 and 3 is needed if the force structure is to be maintained. The planning and management of grade structure, promotion policy, and continuation behavior occur somewhat independently, and it is not clear that a desired pay grade by LOS structure will always be attainable. For example, consider only pay grades E-1 to E-3 and LOS cells 1 to 3. Suppose manpower requirements for pay grades E-1 to E-3 are 200, 200, and 100, respectively. In

addition, suppose that requirements for LOS cells 1 to 3 are 250, 150, and 100, respectively. Then consider the following hypothetical table (table 2), where the starred entries in the matrix exhibit all the pay grade and LOS combinations that may be non-zero.

TABLE 2

PROBLEMATIC FORCE STRUCTURE

	E-1	E-2	E-3	Total
LOS 1	*			250
LOS 2		*	*	150
LOS 3		*	*	100
Total	200	200	100	

One observes an inconsistency in table 2. It is not possible to allocate personnel to the admissible pay grade and LOS combinations in a manner in which pay grade and LOS totals will be maintained. For example, there are 250 personnel with LOS = 1, who by the specified promotion rules must be in pay grade E-1. However, pay grade E-1 is allowed only 200 personnel, a contradiction. Thus, one observes an interdependence between pay grade totals, LOS totals, and promotion policy. They need to be constructed in a manner that does not allow the problems exhibited in table 2 to occur.

This research memorandum addresses the interactions between pay grade structure, longevity, and promotion policy and attains bounds within which force structures are executable. The immediate value of these results is that one is able to describe personnel policies that cannot possibly be successfully executed. The analysis proceeds in purely mathematical terms, and the appendix contains full mathematical detail. The main body of the report avoids mathematical content. Instead, the next section of the report contains an intuitive description of the results followed by some simple applications.

INTUITIVE DESCRIPTION OF RESULTS

Suppose personnel force structure has been defined to the extent that one knows the number of personnel in each pay grade, the longevity distribution of personnel, and the policies that set minimum and maximum longevity levels for each pay grade. One wishes to know whether it is possible to allocate the personnel in an LOS-by-pay-grade matrix in a manner consistent with all the guidelines. It is possible to describe an iterative process (see appendix) that will determine the precise extent to which it is possible to allocate personnel in the desired fashion. The process makes successive approximations to the answer and converges on the best possible allocation of personnel. The situation under which the best possible allocation is not the desired result is directly related to the amount of overlap in LOS cells between different

pay grades. The situation that allows little overlap in LOS cells between successive pay grades is precisely the condition in which a force structure may not be executable.

For example, suppose personnel in pay grade E-4 are not allowed to remain in the Navy if they have more than 8 years of service. Furthermore, suppose that promotion opportunity to E-5 is such that relatively few E-5s have less than 8 years of service. Then, it is quite likely that it will be impossible to attain the desired number of E-4s and E-5s.

These results have an immediate impact on "up or out" policies. It is frequently suggested that one way of saving money is to reduce longevity in pay grades by forcing personnel out of the Navy if they have not been promoted after some certain time in service. The thrust of this memorandum is that, if overdone, such policies may lead to an unexecutable force structure. A certain amount of "slack" is required in the force structure to facilitate effective personnel management. It would be helpful to attain precise limits to the extent of overlap required for a force structure to be executable. Unfortunately, it is not possible to be precise in this area for the following reason. It may be possible to specify a force structure that is attainable under some very precise conditions/forecasts concerning continuation rates. However, continuation behavior varies and does not always follow projections, to say the least. As actual experienced continuation rates vary from projected levels, it will be necessary to vary personnel management policies to successfully attain the desired force structure. The ability to attain the desired force structure is directly related to the slack in the system represented by the permissible overlap in LOS cells between pay grades. If the system does not have enough slack, it will not be feasible to attain the desired force structure. The required amount of slack depends on the experienced variation from predictions and is not known precisely until it happens. The following section describes some examples.

EXAMPLE FORCE STRUCTURES

During the development of force structure the desired inventory should be both attainable and maintainable. Some examples demonstrate how a lack of overlap in allowable longevity between pay grades may lead to problems in both attaining and maintaining a force.

Table 2 is a simplified example. Note the extreme lack of overlap in longevity between E-1s and E-2s: an E-1 can only be in the first year of service, and an E-2 must have completed at least one year of service, so there is no overlap. This is precisely the reason why it is impossible to allocate the personnel in the desired fashion.

From the perspective of force management, then, it is undesirable that the pay-grade-by-LOS matrix split into disjoint components. In

other words, it is undesirable that the LOS cells have no overlap that may be occupied by personnel in different pay grades. The above example was simplified and unrealistic. However, the problems in having little or no overlap between LOS cells that may be occupied by personnel in different pay grades may be further demonstrated. For example, in table 3, the asterisks show permissible pay-grade and LOS combinations. The table again shows a simplified force structure of three grades and three LOS cells, where E-3 personnel may only be in LOS cell 3, etc. Even though there is no longevity overlap between E-3s and E-2s, it is clearly possible to allocate personnel as desired. Table 4 shows one possible allocation.

TABLE 3

ATTAINABLE BUT UNSTABLE FORCE STRUCTURE

	E-1	E-2	E-3	Total
LOS 1	*	*		200
LOS 2	*	*		150
LOS 3			*	100
Total	200	150	100	

TABLE 4

A POSSIBLE ALLOCATION

	E-1	E-2	E-3	Total
LOS 1	150	50		200
LOS 2	50	100		150
LOS 3			100	100
Total	200	150	100	

A problem with the allocation process defined by table 3 is that it is unstable from a practical point of view. The columns in table 3 lead to steady-state continuation rates that are needed to maintain the force structure from one year to the next. In particular, a continuation rate of 100/150 = 0.67 is required between LOS cells 2 and 3. In real life, continuation rates will vary. As a consequence, in some years LOS 3 will have more than 100 personnel and in other years, fewer than 100. However, the number of personnel in pay grade E-3 may equal 100 only if precisely 100 personnel are in LOS cell 3, due to the promotion rules found in the allocation process. In a practical sense, one would like to compensate for the overages or underages in LOS cell 3 by adjusting the number of personnel in LOS cell 2. However, since personnel in LOS cell 2 may not be in pay grade E-3, the problem with E-3 manning cannot be rectified.

A more complex and somewhat more realistic situation is exhibited by the example shown in table 5, where the totality of constraints forces the allocation of personnel to exhibit no overlap in longevity between some pay grades, even though the promotion rules allow for such overlap.

TABLE 5

"APPARENT" OVERLAP

	E-1	E-2	E-3	Total
LOS 1	*	*		200
LOS 2	*	*		150
LOS 3		*	*	100
Total	200	150	100	

Table 5 differs from table 3 in that personnel in LOS cell 3 are allowed to be in pay grade E-2. This apparent flexibility is offset by the requirement to have 100 personnel in pay grade E-3, which forces all personnel in LOS cell 3 to be allocated to pay grade E-3. Hence no personnel from LOS cell 2 are allocated to E-2. Table 5 thus gives rise to an attainable but unstable force structure analogous to the previous example.

The above examples show the type of problems that may arise if there is little overlap between the LOS cells that are admissible to different pay grades: a force structure that is either impossible to attain or very difficult to maintain. The desirable extent of such overlap is determined by other related force structure considerations. For example, the actual allocation of personnel in the pay-grade-by-LOS matrix will have implications on promotion rates, which in turn will affect continuation rates and hence force structure. This memorandum has not attempted to derive exact allocations but has put bounds around the allocation of personnel to pay-grade and LOS combinations.

APPENDIX

AN ALLOCATION PROCESS

APPENDIX

AN ALLOCATION PROCESS

ALLOCATION PROCESSES

The following mathematics is given in a general form. To obtain intuitive insight one should note the following correspondences for the variables described below:

L = the set of LOS cells

l_i = the number of personnel in the i^{th} LOS cell

J = the set of pay grades

p_j = the number of personnel in the j^{th} pay grade

w_{ij} = 1 only if it is allowable for personnel in the j^{th} pay grade to be in the i^{th} LOS cell.

Let L be a set of l elements, divided into I distinct categories $\{L_i | 1 \le i \le I\}$, which have l_i elements, respectively, for $1 \le i \le I$. Thus,

$$\sum_{i=1}^{I} l_i = l .$$

If one divides the l objects into J distinct sets, $\{P_j | 1 \le j \le J\}$, then an allocation process is defined to be a matrix that shows how to allocate the elements of the $\{L_i\}$ into the $\{P_j\}$. In particular, an allocation process is defined by a matrix $A = (a_{ij} | 1 \le i \le I,\ 1 \le j \le J)$, where $a_{ij} \ge 0$ for $1 \le i \le I$, $1 \le j \le J$, and

$$\sum_{j=1}^{J} a_{ij} = 1 \text{ for } 1 \leq i \leq I \ .$$

Thus a_{ij} represents the fraction of L_i that is allocated to P_j.

Suppose one constrains the possible allocations of the $\{L_i\}$ to the $\{P_j\}$ by not allowing each L_i to contribute to every P_J. This rule can be formalized by defining a set of admissible allocations

$\{w_{ij} | 1 \leq i \leq I, \ 1 \leq j \leq J\}$,

where

$$w_{ij} = \begin{cases} 0 \text{ if elements of } L_i \text{ may not be allocated to } P_j \\ 1 \text{ otherwise} \ . \end{cases}$$

It is evident that $a_{ij} \neq 0$ only if $w_{ij} = 1$. Further suppose that it is desired that P_j have p_j elements for $1 \leq j \leq J$. This may not always be possible. Trivially, if

$$\sum_{i=1}^{I} l_i \neq \sum_{j=1}^{J} p_i \ ,$$

then one cannot allocate the elements of L as desired. At a less trivial level, the number of elements allocated to P_j is given by

$$\sum_{i=1}^{I} a_{ij} l_i \ .$$

If "too many" of the a_{ij} are constrained equal to zero by the admissible allocation rules $\{w_{ij}\}$, then it may not be possible to select values for the $\{a_{ij} | 1 \le i \le I\}$ such that

$$p_j = \sum_{i=1}^{I} a_{ij} l_i \ .$$

For given values of the l_i for $1 \le i \le I$, the p_j for $1 \le j \le J$ and the admissible allocation rules w_{ij} for $1 \le i \le I$, $1 \le j \le J$, it is of interest to determine whether or not one can allocate the L_i as desired. If it is not possible to carry out the desired allocation, it is also of interest to determine how "closely" one can attain the desired goal. This situation may be fully described by use of the algorithm that is developed below.[1]

ALLOCATION ALGORITHM

For a given allocation process, the number of elements of the $\{L_i\}$ allocated to P_j is given by

$$\sum_{i=1}^{I} a_{ij} l_i \ .$$

1. This algorithm is described in CNA Research Contribution 533, *National Manpower Inventory Final Report*, by Aline Quester et al., September 1985. There it was used to "crosswalk" individuals between civilian and military jobs. That document does not prove or fully describe the properties of the algorithm, which are shown herein.

Let

$$R_j = p_j / \sum_{i=1}^{I} a_{ij} l_i \ .$$

Then R_j measures how "closely" the allocation process matched the desired goal of

$$p_j = \sum_{i=1}^{I} a_{ij} l_i \ .$$

$R_j = 1$ precisely when the allocation goal is met, and the further R_j is from unity, then the further the allocation is from attaining its goal. An efficient allocation scheme is defined as one in which the R_j are as close together as possible under the constraints given by the l_i, p_j, and w_{ij}, for $1 \leq i \leq I$, $1 \leq j \leq J$. As is shown below, an efficient allocation scheme may always be attained by application of an algorithm that successively amends an initial guess at an allocation process until it converges into an efficient allocation scheme.

Let $A^{(0)}$ represent an initial guess at an allocation scheme. So, $A^{(0)} = \{a_{ij}^{(0)} | 1 \leq i \leq I,\ 1 \leq j \leq J\}$ is such that

$$a_{ij}^{(0)} \geq 0 \text{ for } 1 \leq i \leq I,\ 1 \leq j \leq J \ ,$$

$$\sum_{j=1}^{J} a_{ij}^{(0)} = 1 \quad \text{for} \quad 1 \leq j \leq J \ , \text{ and}$$

$$a_{ij}^{(0)} \neq 0 \text{ implies } w_{ij} = 1 \text{ for } 1 \leq i \leq I,\ 1 \leq j \leq J \ .$$

Define $$R_j^{(n)} = p_j / \sum_{i=1}^{I} a_{ij}^{(n)} l_i \qquad \ldots \qquad \text{(A-1)}$$

and $$a_{ij}^{(n+1)} = a_{ij}^{(n)} \cdot R_j^{(n)} / \sum_{k=1}^{J} a_{ik}^{(n)} R_k^{(n)} \qquad \ldots . \qquad \text{(A-2)}$$

Then it is claimed that this iterative process converges to an efficient allocation scheme. First, one needs to verify that

$$A^{(n+1)} = (a_{ij}^{(n+1)})$$

is indeed an allocation scheme, even if it is not efficient. Evidently, the $a_{ij}^{(n+1)}$ are all nonnegative numbers. It suffices to show that

$$\sum_{j=1}^{J} a_{ij}^{(n+1)} = 1 \quad \text{for} \quad 1 \le i \le I .$$

From equation A-2 one has

$$\sum_{j=1}^{J} a_{ij}^{(n+1)} = \sum_{j=1}^{J} a_{ij}^{(n)} R_j^{(n)} / \sum_{k=1}^{J} a_{ik}^{(n)} R_k^{(n)} = 1 .$$

So, $A^{(n+1)}$ does indeed define an allocation scheme.

Before one can prove that the above process converges, it is necessary to consider the matrix $A^{(0)}$ in orthogonal blocks. By suitably rearranging the rows and columns of $A^{(0)}$, one can represent $A^{(0)}$ by

$$A^{(0)} = \begin{pmatrix} A_1 & & & 0 \\ \cdot & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix} ,$$

where the A_i are submatrices and all the elements of $A^{(0)}$ are zero outside of the A_i. Moreover, one can continue this process until the A_i are indecomposable orthogonal block matrices. With A_i being indecomposable one can trace a "path" among nonzero elements of A_i, that is, if a_{ij}, $a_{gh} \varepsilon A_i$, then there exists a sequence of nonzero elements of A_i,

$$a_{i_1 j_1}, a_{i_1 j_2}, a_{i_2 j_2}, a_{i_2 j_3}, \cdots a_{i_n j_n}$$

such that

$$a_{ij} = a_{i_1 j_1} \text{ and } a_{i_n j_n} = a_{gh} .$$

If this were not so, then A_i would be decomposable into two distinct orthogonal blocks, containing a_{ij} and a_{gh}, respectively. One may now state the following propositions:

Proposition (1): $R_j^{(n)}$ converges to a limit R_j, say, for $1 \leq j \leq J$

Proposition (2): $A^{(n)} = (a_j^{(n)})$ converges to a limit A, say, though different values of A may produce the same R_j

Proposition (3): If $a_{ij} \neq 0$, let a_{ij} belong to the indecomposable block A_k , say, of the limiting matrix A. Then,

$$R_j = \sum_{t\varepsilon T} p_t / \sum_{s\varepsilon S} l_s ,$$

where S is the set of rows of A_k, and T is the set of columns of A_k. Note this implies that R_j is invariant for all j that correspond to the same indecomposable block.

Proposition (4): Consider another allocation $(b_{ij}^{(n)})$, and let

$$Q_j^{(n)} = p_j / \sum_{i=1}^{I} b_{ij}^{(n)} l_i .$$

Suppose $Q_j^{(n)}$ converges to Q_j. If $a_{ij}^{(0)} \neq 0$ if and only if $b_j^{(0)} \neq 0$ for $1 \leq i \leq I$, $1 \leq j \leq J$, then $Q_j = R_j$ for $1 \leq j \leq J$. In the proof, propositions 1 to 3 are proved together. Let

$$R_c^{(n)} = \max_k \{R_k^{(n)}\} \text{ and let } R_d^{(n+1)} = \max_k \{R_k^{(n+1)}\} .$$

One needs to show that

$$R_d^{(n+1)} \leq R_c^{(n)} .$$

This, taken in conjunction with the fact that the $R_j^{(n)}$ are bounded below by

$$p_j \Big/ \sum_{i=1}^{I} l_i$$

shows that the maximal values of the $\{R_k^{(n)}\}$ converge.

Suppose

$$R_c^{(n)}/R_d^{(n)} = \alpha \ . \tag{A-3}$$

Now,

$$R_d^{(n+1)} = p_d \Big/ \sum_{i=1}^{I} a_{id}^{(n+1)} l_i \ , \tag{A-4}$$

and

$$a_{id}^{(n+1)} = a_{id}^{(n)} R_d^{(n)} \Big/ \sum_{j=1}^{J} a_{ij}^{(n)} R_j^{(n)} \tag{A-5}$$

from equations A-1 and A-2. Substituting A-5 into A-4 gives

$$R_d^{(n+1)} = p_d \Big/ \left(\sum_i a_{id}^{(n)} R_d^{(n)} l_i \Big/ \sum_j a_{ij}^{(n)} R_j^{(n)} \right) \ .$$

From A-3, this implies

$$R_d^{(n+1)} = \alpha\ p_d \cdot \left(\sum_i a_{id}^{(n)} R_c^{(n)} 1_i / \sum_j a_{ij}^{(n)} R_j^{(n)}\right) .$$

Now,

$$R_c^{(n)} / \sum_j a_{ij}^{(n)} R_j^{(n)} \geq 1 \text{ since } R_c^{(n)} = \max_k \left(R_k^{(n)}\right) .$$

So,

$$R_d^{(n+1)} \leq \alpha\ p_d \sum_i a_{id}^{(n)} 1_i = \alpha\ R_d^{(n)} = R_c^{(n)} ,$$

as desired. So, the maximal value of $\{R_j^{(n)}\}$ converges as $n \rightarrow \infty$. It remains to show that the smaller values of the $\{R_j^{(n)}\}$ also converge as $n \rightarrow \infty$. In order to accomplish this, one needs to also consider and demonstrate the convergence of the $(a_{ij}^{(n)})$.

Consider

$$\begin{aligned} a_{id}^{(n+2)} &= a_{id}^{(n+1)} R_d^{(n+1)} / \sum_j a_{ij}^{(n+1)} R_j^{(n+1)} \\ &\leq a_{id}^{(n+1)} R_d^{(n+1)} / \sum_j a_{ij}^{(n+1)} R_d^{(n+1)}, \text{ since } R_d^{(n+1)} \text{ is maximal} \\ &= a_{id}^{(n+1)} . \end{aligned}$$

So,

$$a_{id}^{(n+2)} \le a_{id}^{(n+1)} \text{ for } 1 \le i \le I .$$

Thus,

$$R_d^{(n+2)} = p_d / \sum_i a_{id}^{(n+1)} l_i$$

$$\le p_d / \sum_i a_{id}^{(n)} l_i$$

$$= R_d^{(n+1)} .$$

In the limiting situation one has that $R_d^{(n+2)}$ is arbitrarily close to $R_d^{(n+1)}$, the $a_{id}^{(n+2)}$ are arbitrarily close to the $a_{id}^{(n+1)}$ for each value of i, that is, the $a_{id}^{(n)}$ have also converged for $1 \le i \le I$.

However, as the values of

$$a_{id}^{(n+1)}, a_{id}^{(n+2)}, \ldots,$$

converge one has

$$R_d^{(n+1)} / \sum_j a_{ij}^{(n+1)} R_j^{(n+1)}$$

converging to 1. Since $R_d^{(n+1)}$ is maximal, this means that

$$a_{id}^{(n+1)} \neq 0 \neq a_{ij}^{(n+1)}$$

implies $R_j^{(n+1)}$ is arbitrarily close to $R_d^{(n+1)}$.

So, if

$$\lim_{n\to\infty} a_{id}^{(n)} \neq 0, \text{ either } \lim_{n\to\infty} a_{ij}^{(n)} = 0$$

or

$$\lim_{n\to\infty} R_j^{(n)} = \lim_{n\to\infty} R_d^{(n)} .$$

If

$$\lim_{n\to\infty} R_j^{(n)} = \lim_{n\to\infty} R_d^{(n)} ,$$

then, by the same argument as applied to the $a_{id}^{(n)}$ for $1 \le i \le I$, [illegible] as that the $a_{ij}^{(n)}$ converge for $1 \le i \le I$. One can now repeat this process for any j

where

$$\lim_{n\to\infty} R_j^{(n)} = \lim_{n\to\infty} R_d^{(n)},$$

by considering those k where

$$\lim_{n\to\infty} a_{kj}^{(n)} \neq 0$$

and show that either

$$\lim_{n\to\infty} a_{km}^{(n)} = 0 \text{ or } \lim_{n\to\infty} R_j^{(n)} = \lim_{n\to\infty} R_m^{(n)}.$$

This process can be continued for all the columns of $(a_{ij}^{(n)})$ until one has produced an indecomposable block inside $(a_{ij}^{(n)})$, with columns T and rows S such that

$$\lim_{n\to\infty} R_k^{(n)} = \lim_{n\to\infty} R_j^{(n)} \text{ for every } k,j \in T .$$

The convergence of the $R_j^{(n)}$ and $a_{ij}^{(n)}$ inside the indecomposable block has been demonstrated. It remains to consider the elements outside of the block.

If the above process included all of the rows and columns of $(a_{ij}^{(n)})$ then the convergence of the $R_j^{(n)}$ and $a_{ij}^{(n)}$ has been demonstrated for all values of i and j. If the process produced an indecomposable

block inside $(a_{ij}^{(n)})$, then the proof follows by a simple inductive argument. By assumption, the matrix $(a_{ij}^{(n)})$ decomposes into two orthogonal blocks, that is,

$$a_{ij}^{(n)} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

say, with

$$a_{id}^{(n)} \varepsilon A_1 .$$

Then, by induction on the dimension of the matrix, convergence inside A_2 is demonstrated. So, propositions 1 and 2 are proved.

In order to complete the proof of proposition 3 one needs to demonstrate

$$R_j = \lim_{n\to\infty} R_j^{(n)} = \sum_{s\varepsilon S} l_s \Big/ \sum_{t\varepsilon T} p_t .$$

Let

$$\lim_{n\to\infty} a_{ij}^{(n)} = a_{ij} .$$

Suppose $\lambda = R_j$ for $j \varepsilon T$. Then

$$\lambda = R_j = p_j \Big/ \sum_{i=1}^{I} a_{ij} l_i ,$$

by definition. However, the a_{ij} are zero outside of S.

So

$$\lambda = R_j = p_j \Big/ \sum_{s \varepsilon S} a_{sj} l_s .$$

Hence

$$p_j/\lambda = \sum_{s \varepsilon S} a_{sj} l_s .$$

Thus,

$$\sum_{t \varepsilon T} p_j/\lambda = \sum_{t \varepsilon T} \sum_{s \varepsilon S} a_{st} l_s = \sum_{s \varepsilon S} \sum_{t \varepsilon T} a_{st} l_s .$$

But

$$\sum_{t \varepsilon T} a_{st} = 1 .$$

Thus,

$$\sum_{t \varepsilon T} p_t/\lambda = \sum_{s \varepsilon S} l_s ;$$

hence,

$$R_j = \lambda = \sum_{t \varepsilon T} p_t \Big/ \sum_{s \varepsilon S} l_s \, ,$$

as desired.

It now remains to consider the proof of proposition 4. So, consider another initial allocation $(b_{ij}^{(0)})$, where $a_{ij}^{(0)} \neq 0$ if and only if $b_{ij}^{(0)} \neq 0$. Further suppose that

$$Q_j^{(n)} = p_j \Big/ \sum_{i=1}^{I} b_{ij}^{(n)} l_i$$

converges to Q_j, and $b_{ij}^{(n)}$ converges to b_{ij} for $1 \leq i \leq I$, $1 \leq j \leq J$. One wants to prove $Q_j = R_j$ for $1 \leq j \leq J$. Suppose this is not so. Then there exists j such that $Q_j \neq R_j$. Without loss of generality one may assume the j are ordered such that $R_j \leq R_k$ if $j \leq k$. Since the (a_{ij}) split into orthogonal blocks, inductive reasoning shows that it suffices to assume $Q_J \neq R_J$. Again without loss of generality one may assume that $R_J > Q_J$. Now $R_J > Q_J$ implies there exists an i such that either (a) $a_{iJ} = 0$ and $b_{iJ} \neq 0$ or (b) $a_{iJ} \neq 0$ and $b_{iJ} = 0$. If this were not so, then by proposition 3, R_J would equal Q_J. Each possibility is considered in turn and shown to be impossible.

Case (a): $a_{iJ} = 0$ and $b_{iJ} \neq 0$.

Now $b_{iJ} \neq 0$ implies $b_{ij}^{(0)} \neq 0$ implies $a_{iJ}^{(0)} \neq 0$ by assumption. At some stage n, R_J becomes maximal. Since n is finite $a_{iJ}^{(n)} \neq 0$.

Now

$$a_{iJ}^{(n+1)} = a_{iJ}^{(n)} R_J^{(n)} / \sum_{J=1}^{J} a_{ij}^{(n)} R_j^{(n)} .$$

Since

$$R_J^{(n)} = \max_j \, (R_j^{(n)}), \text{ one has } R_J^{(n)} / \sum_{j=1}^{J} a_{ij}^{(n)} R_j^{(n)} \geq 1 .$$

Hence

$$a_{iJ}^{(n+1)} \geq a_{iJ}^{(n)} ,$$

and this will happen whenever $R_J^{(n)}$ is maximal. So

$$\lim_{n \to \infty} a_{iJ}^{(n)} \neq 0 ,$$

a contradiction.

Case (b): $a_{iJ} \neq 0$ and $b_{iJ} = 0$.

Now a_{iJ} belongs to an indecomposable block with columns T and rows S. Suppose b_{iJ} belongs to an indecomposable block with columns V and rows U. Case (a) shows that $b_{iJ} \neq 0$ implies $a_{iJ} \neq 0$. All of the columns, k, of the indecomposable block containing b_{ij} have the same

value of $Q_k = Q_J$. Hence the inference $b_{iJ} \neq 0$ implies $a_{iJ} \neq 0$ can be expanded to cover all of the block containing b_{iJ}. This means that $T \supseteq V$ and $S \supseteq U$. It is shown that $T = V$. Suppose otherwise, then there exists $t \in T$ such that $b_{ut} = 0$ for every $u \in U$ (this is because the b_{ij} form orthogonal blocks). Now for any i where $1 \leq i \leq I$ one obviously has one of the following:

(i) $i \in U$

(ii) $i \in S/U$, assuming $S \neq U$

(iii) $i \notin S$.

From the preceding statement $i \in U$ implies $b_{it} = 0$. Also, $i \in S/U$ implies $b_{it} = 0$ by the definition of the difference between S and U.

Finally, $i \in S$ implies $b_{it} = 0$, otherwise one has the situation of case 1 where $a_{it} = 0$ and $b \neq 0$. Thus $b_{it} = 0$ for $1 \leq i \leq I$, which is impossible. (If $b_{it}^{(n)}$ are all very close to zero for $1 \leq i \leq I$, then $R_t^{(n)}$ would be very large and presumably $R_t^{(n)}$ would be $\max_k (R_k^{(n)})$. As shown above, the maximal values are monotonically decreasing. Hence it is impossible for $\lim_{n\to\infty} b_{it}^{(n)} = 0$ for $1 \leq i \leq I$.) So $T = V$. Since $S \supseteq U$, this means

$$\begin{aligned} R_J &= \sum_{t\in T} p_t / \sum_{s\in S} l_s \\ &\leq \sum_{t\in T} p_t / \sum_{s\in U} l_s \\ &= \sum_{t\in V} p_t / \sum_{s\in U} l_s = Q_j \ , \end{aligned}$$

contradicting the assumption that $R_J > Q_J$ and completing the proof.